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TONGUES IN PARAMETRIC RESONANCE

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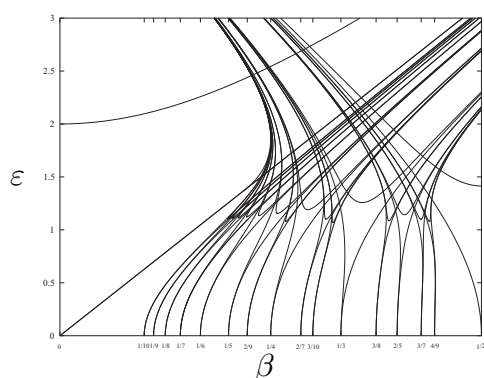


Figure 1. Resonance tongues in the Arnold family [Broer, Simó and Tatjer, 1998].

Abstract

Resonance is the interaction between various oscillating subsystems of a given family of dynamical systems, and as such is part of the bifurcation scenario of the given family. Aspects of the bifurcation geometry are the interest of this paper. In particular we deal with so-called instability pockets where we focuss on an example with discontinuities.

1 Introduction

Resonance in a dynamical system is an interaction between two or more oscillating subsystems, the frequencies of which enjoy a rational relationship. To describe resonance we need dynamical systems that depend on parameters and resonance then is part of a bifurcation scenario. Often the resonance dynamics corresponds to phenomena associated to hyperbolicity, like phase locking and synchronisation or to instability.

1.1 Hopf-Neïmark-Sacker

In the following we present examples where the resonant dynamics in the parameter space occurs in ‘tongues’. We start considering the Arnold family of

circle maps

$$P_{\beta,\varepsilon} : \mathbb{S}^1 \rightarrow \mathbb{S}^1; \quad x \mapsto x + 2\pi\beta + \varepsilon \sin x,$$

where $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ denotes the circle. In Figure 1 we depict the (β, ε) -plane of parameters. Here (Arnold) tongues emanate from all ‘resonances’ $(\beta, \varepsilon) = (\frac{p}{q}, 0)$, with $p, q \in \mathbb{N}$, into the half plane $\varepsilon > 0$, see [Arnold, 1983, Broer, Simó and Tatjer, 1998]. Inside the tongues the circle dynamics is Morse-Smale, with an alteration of attracting and repelling periodic points with rotation number $\frac{p}{q}$. In the complement of the tongues the circle dynamics is quasi-periodic.

Remark. This kind of scenario also occurs in the Hopf-Neïmark-Sacker bifurcation [Arnold, 1983, Ciocci, Litvak-Hinenzon and Broer, 2005]. Here the eigenvalues of a fixed point of a map cross the complex unit circle on an arc that does not contain any of the points $e^{\pm 2\pi ki}$, $0 \leq k \leq 4$, associated to strong resonance [Takens, 1974]. For a discussion on the geometry of the weak resonance tongues see [Broer, Golubitsky and Vegter, 2003, Broer, Golubitsky and Vegter, 2007, Broer and Vegter, 2008] and references therein.

1.2 Parametric resonance

Another class of examples of resonance concern the parametrically forced pendulum

$$\ddot{x} + (a + b p(t)) \sin x = 0, \quad (1)$$

where p is periodic in the time t . Here we consider the (a, b) -plane of parameters and observe the emanance of resonance tongues from the ‘resonances’ $(a, b) = (\frac{1}{4}n^2, 0)$ into the half planes $b \neq 0$. To explain this note that (1) always admits the trivial periodic solution $x = 0 = \dot{x}$. The tongues precisely contain the parameter points for which this trivial solution is unstable. For a

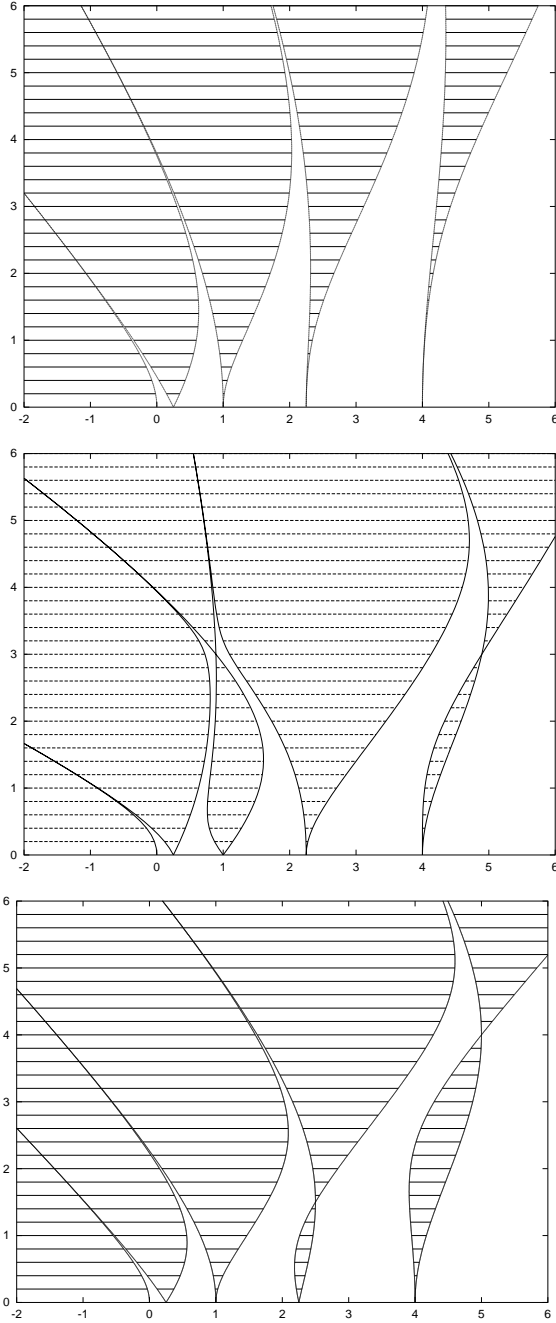


Figure 2. Resonance tongues (shaded). Top: the classical Mathieu case $p(t) = \cos t$. Middle: a modified Mathieu case $p(t) = \cos t + c \cos(2t)$ with $c = 1.5$. Bottom: square Hill's case [Broer and Levi, 1995, Broer and Simó, 2000].

discussion see [Broer, Hoveijn, Van Noort and Vegter, 1999, Broer and Vegter, 1992] and references therein.

This so-called stability diagram already is determined by the linear equation

$$\ddot{x} + (a + b p(t))x = 0, \quad (2)$$

(the so-called Mathieu equation), which is a special kind of Hill's equation, see [Broer and Levi, 1995,

Broer, Simó and Puig, 2003, Broer and Simó, 2000] and references therein. In Figure 2 part of the stability diagram is shown for the cases $p(t) = \cos t$ (the Mathieu case), $p(t) = \cos t + c \cos 2t$ or $p(t) = \text{sgn } \cos t$ (the square Hill case).

Observe that in Figure 2, the tongues in the middle and bottom figure exhibit 'instability pockets' [Broer and Levi, 1995, Broer and Simó, 2000]. Also it is known that in the classical Mathieu case, see the top of Figure 2, no such pockets occur [Meixner and Schäfer, 1954]. We like to mention that also in the Hopf-Neimark-Sacker case modifications of tongues occur, often described as 'flames' [Broer, Golubitsky and Vegter, 2003, Broer, Golubitsky and Vegter, 2007, Broer and Vegter, 2008] and references therein.

The present aim is to describe this stability diagram in the latter, square Hill case. Compare with [Broer and Levi, 1995, Broer and Simó, 2000].

2 Preliminaries

For differential equations like (1) and (2) consider the Poincaré map

$$P_{a,b} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (3)$$

defined by following the 3D phase flow over a full period [Arnold, 1980, Arnold, 1983, Broer and Levi, 1995]. Since the systems at hand are conservative, the maps are area preserving. Moreover in the linear case of (2) the map $P_{a,b}$ also is linear. This means that $P_{a,b} \in Sp(1)$, the 3D symplectic group of 2×2 matrices. Stability of the trivial periodic solution is equivalent to

$$\text{spec } P_{a,b} \subset \mathbb{S}^1 \setminus \{+1, -1\},$$

or equivalently $|\text{tr } P_{a,b}| < 2$.

We only consider functions $p = p(t)$ that are even, i.e., with $p(-t) = p(t)$, which amounts to reversibility of (1) and (2). In that case the linear Poincaré maps $P_{a,b}$ are confined to the 1-sheeted hyperboloid

$$SR(1) = \left(\begin{array}{cc} u & v + w \\ -v + w & u \end{array} \right), \text{ with} \\ u^2 + v^2 = 1 + w^2, \quad (4)$$

compare with [Broer and Levi, 1995]. Now $P_{a,b}$ is stable if and only if $|u| < 1$. If we also consider the polar angle $\varphi = \arg(u + iv)$, the covering $\widetilde{SR(1)}$ is a plane with coordinates (φ, w) . Here the stability boundaries $|u| = 1$ are given by

$$w = \pm \tan(\varphi - k\pi), k = 0, 1, 2, \dots \quad (5)$$

In this way we obtain Hill's map

$$H : \mathbb{R}^2 \rightarrow SR(1), (a, b) \mapsto P_{a,b}, \quad (6)$$

that wraps and folds the parameter plane around the hyperboloid. In [Broer and Levi, 1995, Broer, Simó and Puig, 2003, Broer and Simó, 2000] the map H is investigated with the help of Singularity Theory. In the sequel we also will consider the corresponding map $\tilde{H} : \mathbb{R}^2 \rightarrow \widetilde{SR(1)}$. To find the tongue boundaries in the (a, b) -plane we have to pull back the curves (5) along \tilde{H} .

3 Square Hill's equation

In this section we confine ourselves to the square Hill equation

$$\ddot{x} + (a + b \operatorname{sgn} \cos t)x = 0, \quad (7)$$

i.e. the equation (2) with $p(t) = \operatorname{sgn} \cos t$. Main fact is that the function $p(t) = \operatorname{sgn} \cos t$ is piecewise constant. On each of the pieces the square Hill equation (7) is linear. We summarize the results of [Broer and Levi, 1995], that also recover asymptotic estimates in [Arnold, 1980]. First of all it is easy to see that the whole diagram is symmetric under reflection in the a -axis.

3.1 Introduction

A rescaling of time in (7) allows a reparametrization of the 'interesting' part of the (a, b) -plane. In this way the Poincaré map can be decomposed in a convenient way, involving a Euclidean rotation. Indeed, putting

$$\tau = \frac{T}{2\pi}t,$$

where T is another parameter, turns equation (7) into

$$x'' + \left(\frac{2\pi}{T}\right)^2 \left(a \pm b \operatorname{sgn} \cos \left(\frac{2\pi}{T}\tau\right)\right)x = 0.$$

We now specify T such that

$$\left(\frac{2\pi}{T}\right)^2 (a + b) = 1 \quad (8)$$

and introduce a fourth parameter B by

$$B = \left(\frac{2\pi}{T}\right)^2 (a - b). \quad (9)$$

We claim that

$$(a, b) \mapsto (B, T) = \left(\frac{a-b}{a+b}, 2\pi\sqrt{a+b}\right) \quad (10)$$

is a (diffeomorphic) reparametrization when (a, b) is restricted to the domain $Q = \{(a, b) \in \mathbb{R}^2 \mid a > |b|\}$, corresponding to $\{(B, T) \in \mathbb{R}^2 \mid B > 0, T > 0\}$. The quadrant Q contains all instability pockets.

For simplicity we rename τ by t again and rewrite (7) to

$$\ddot{x} + p(t)x = 0, \quad p(t) = \begin{cases} 1 & \text{if } \cos\left(\frac{2\pi}{T}t\right) > 0 \\ B & \text{otherwise.} \end{cases} \quad (11)$$

We now turn our attention to the reparametrized version $H : (B, T) \mapsto P_{B,T}$ of Hill's map.

It turns out that in this setting the Poincaré map $P \in SR(1)$ can be composed as

$$P = P_1 \circ P_2 \circ P_1, \quad (12)$$

with

$$P_1 = R_{\frac{1}{4}T},$$

where R_α denotes the Euclidean rotation over an angle α , and where

$$P_2 = \begin{pmatrix} \cos\left(\frac{1}{2}T\sqrt{B}\right) & \frac{1}{\sqrt{B}}\sin\left(\frac{1}{2}T\sqrt{B}\right) \\ -\sqrt{B}\sin\left(\frac{1}{2}T\sqrt{B}\right) & \cos\left(\frac{1}{2}T\sqrt{B}\right) \end{pmatrix}.$$

We now investigate Hill's map $\tilde{H} : Q \rightarrow \widetilde{SR(1)} = \{\varphi, w\}$, recalling that $\varphi = \arg(u + iv)$. In the space $\mathbb{R}^3 = \{\varphi, w, T\}$ consider the surface S parametrized by $\Phi : (B, T) \mapsto (\varphi, w, T)$ as follows:

$$\begin{aligned} u &= \cos\left(\frac{1}{2}T\sqrt{B}\right), \\ v &= \frac{1}{2}\left(\sqrt{B} + \frac{1}{\sqrt{B}}\right)\sin\left(\frac{1}{2}T\sqrt{B}\right), \\ w &= \frac{1}{2}\left(\sqrt{B} - \frac{1}{\sqrt{B}}\right)\sin\left(\frac{1}{2}T\sqrt{B}\right). \end{aligned} \quad (13)$$

Also consider the skew projection Π given by

$$(\varphi, w, T) \mapsto (\varphi + \frac{1}{2}T, w). \quad (14)$$

One can show that

$$\tilde{H} = \Pi \circ \Phi. \quad (15)$$

Writing P_2 in the coordinates (u, v, w) of (4) gives (14).

Remark. To compute the product (12), we observe the simple but important relation: If

$$\begin{pmatrix} U & V+W \\ -V+W & U \end{pmatrix} = \quad (16)$$

$$R_\alpha \begin{pmatrix} u & v+w \\ -v+w & u \end{pmatrix} R_\alpha,$$

then

$$\begin{pmatrix} U \\ V \end{pmatrix} = R_{2\alpha} \begin{pmatrix} u \\ v \end{pmatrix}, \quad W = w. \quad (17)$$

Applying this remark to (12) we conclude that the (U, V, W) -coordinates of P are given by (17) with $2\alpha = \frac{1}{2}T$ such that $\arg(U + iV) = \varphi + \frac{1}{2}T$. This proves (14).

3.2 Conclusions

In summary, the tongue boundaries in the parameter plane are the pull back under \tilde{H} of the curves (5) along \tilde{H} . Using the decomposition (15)

$$\tilde{H} : Q \xrightarrow{\Phi} S \subset \mathbb{R}^3 \xrightarrow{\Pi} \widetilde{SR(1)}, \quad (18)$$

we have to pull back the curves (5) along Π , so to obtain counterpart curves inside the surface S , and next to compute the inverse images of these under Φ . In this way we directly recover the sharpness of the resonance tongues [Arnold, 1980]:

$$\begin{aligned} a &= \frac{1}{4}n^2 \pm \frac{2}{n\pi}b + o(b) \text{ for odd } n, \\ a &= \frac{1}{4}n^2 \pm \frac{4}{n^2}b^2 + o(b^2) \text{ for even } n. \end{aligned} \quad (19)$$

We further summarize the computations as follows, compare with Figure 2 (bottom).

Theorem [Broer and Levi, 1995] **[Properties of the stability diagram]** *The two boundaries of the stability regions which meet at the n th resonance have exactly n points of intersection, counting multiplicity. For n odd, n of these intersections are transversal, yielding n distinct points and at least $n + 1$ instability pockets.*

For n even the two boundary curves have a nondegenerate quadratic tangency at the resonance, compare with (19), while the remaining $n - 2$ intersections are transversal, yielding at least n instability pockets.

Remarks.

- i. The existence of n intersections (counting multiplicity) between the boundaries of the n th instability persists under small reversible perturbations of square Hill's equation.
- ii. The lift \tilde{H} still fails to be 1 : 1 since it has infinitely many fold lines, all 'due to' the projection Π .
- iii. Square Hill's equation generates a dynamical system with discontinuities in the generalized phase space $\mathbb{R}^3 = \{x, \dot{x}, t\}$. One question is how this kind of discontinuity can be incorporated in the Filippov formalism [Leine and Nijmeijer, 2004].

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